5.2 – Diagonalization

Due Fri

Definition: A square matrix A is said to be **diagonalizable** if it is similar to some diagonal matrix; that is, if there exists an invertible matrix P such that $P^{-1}AP$ is diagonal. In this case, the matrix P is said to **diagonalize** A.

Theorem 5.2.1 If A is an $n \times n$ matrix, then A is diagonalizable if and only if A has n linearly independent eigenvectors.

#5 Find a matrix P that diagonalizes A, and check your work by computing $P^{-1}AP$.

$$A = \begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix}$$

$$\det(\lambda I - A) = 0 \Rightarrow \begin{vmatrix} \lambda-1 & 0 \\ -6 & \lambda+1 \end{vmatrix} = 0$$

$$\Rightarrow \lambda = \pm 1$$

eigenspace basis vector

$$\lambda_1 = -1 \quad \begin{bmatrix} -2 & 0 \\ -6 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \to x_1 = 0 \quad \vec{x}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\lambda_2 = 1 \quad \begin{bmatrix} 0 & 0 \\ -6 & 2 \end{bmatrix} \to \begin{bmatrix} 3 & -1 \\ 0 & 0 \end{bmatrix} \xrightarrow{x_1 = \frac{1}{3}x_2} \vec{x}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Notational change: $\vec{p}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \vec{p}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

Form $P = \left[\vec{p}_1 \mid \vec{p}_2 \right] = \begin{bmatrix} 0 & 1 \\ 1 & 3 \end{bmatrix}$

$$P^{-1} = \begin{bmatrix} -3 & 1 \\ 1 & 0 \end{bmatrix}$$

Compute: $P^{-1}AP = \begin{bmatrix} -3 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 3 \end{bmatrix}$

$$= \begin{bmatrix} -3 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = D$$

diagonal matrix

$P^{-1}AP = D$ results in a diagonal matrix if A is diagonalizable.

Notes:
- Diagonal entries of $P^{-1}AP$ are eigenvalues.
- Columns of P are their respective eigenvectors
- → The order of these is connected.

What makes this work?

$$P^{-1}AP = D \Rightarrow AP = PD$$

Here,

$$\begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\left[A\vec{p}_1 \mid A\vec{p}_2 \right] = \left[\lambda_1 \vec{p}_1 \mid \lambda_2 \vec{p}_2 \right]$$

$$\Rightarrow A\vec{p}_1 = \lambda_1 \vec{p}_1 , \quad A\vec{p}_2 = \lambda_2 \vec{p}_2$$

Those satisfy $A\vec{x} = \lambda \vec{x}$.

#17 Compute the matrix A^{10} by first diagonalizing A.

$$A = \begin{bmatrix} 0 & 3 \\ 2 & -1 \end{bmatrix}$$

Note that if $P^{-1}AP = D$, then $A = PDP^{-1}$.

Then $A^k = (PDP^{-1})(PDP^{-1}) \cdots (PDP^{-1})(PDP^{-1})$

$$= PD^kP^{-1}$$

Useful because if $D = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \; \lambda_n \end{bmatrix}$, $D^k = \begin{bmatrix} \lambda_1^k & & 0 \\ & \lambda_2^k & \\ 0 & & \ddots \; \lambda_n^k \end{bmatrix}$

$$\begin{vmatrix} \lambda & -3 \\ -2 & \lambda+1 \end{vmatrix} = 0 \Rightarrow \lambda^2 + \lambda - 6 = 0 \Rightarrow \lambda = -3, 2$$

$\lambda = -3$ $\begin{bmatrix} -3 & -3 \\ -2 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \to \vec{p}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\lambda = 2$ $\begin{bmatrix} 2 & -3 \\ -2 & 3 \end{bmatrix} \to \begin{bmatrix} 2 & -3 \\ 0 & 0 \end{bmatrix} \to \vec{p}_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

$$P = \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix} \Rightarrow P^{-1} = \frac{1}{5}\begin{bmatrix} 2 & -3 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2/5 & -3/5 \\ 1/5 & 1/5 \end{bmatrix}$$

$$D = \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix}$$

$$A^{10} = PD^{10}P^{-1} = \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3^{10} & 0 \\ 0 & 2^{10} \end{bmatrix}\begin{bmatrix} 2/5 & -3/5 \\ 1/5 & 1/5 \end{bmatrix}$$

$$A^{10} = \begin{bmatrix} 24,234 & -34,815 \\ -23,210 & 35,839 \end{bmatrix}$$

Theorem 5.2.3 If k is a positive integer, λ is an eigenvalue of a matrix A, and $\mathbf{x}$ is a corresponding eigenvector, then λ^k is an eigenvalue of A^k and $\mathbf{x}$ is a corresponding eigenvector.

(not a proof for all k)

Consider $A\vec{x} = \lambda\vec{x} \Rightarrow A^2\vec{x} = A\lambda\vec{x}$

$= \lambda A\vec{x} = \lambda(\lambda\vec{x}) = \lambda^2\vec{x}$

$A^2\vec{x} = \lambda^2\vec{x}$

#9 Let $A = \begin{bmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{bmatrix}$.

$P^{-1}AP$ — square

3x3 3x3 → 3 columns — not invertible

$[\vec{p}_1 \; \vec{p}_2 \; \vec{p}_1]$

a. Find the eigenvalues of A.

b. For each eigenvalue λ, find the rank of the matrix $\lambda I - A$.

c. Is A diagonalizable? Justify your conclusion.

$\begin{vmatrix} \lambda-4 & 0 & -1 \\ -2 & \lambda-3 & -2 \\ -1 & 0 & \lambda-4 \end{vmatrix} = 0 \Rightarrow (\lambda-3)\begin{vmatrix} \lambda-4 & -1 \\ -1 & \lambda-4 \end{vmatrix} = 0$

$(\lambda-3)\left[(\lambda-4)^2-1\right] = 0$

$\lambda^2 - 8\lambda + 16 - 1$

$(\lambda-3)(\lambda-3)(\lambda-5) = 0 \Rightarrow \lambda = 3$ (alg. mult. 2), 5

$\lambda_1 = 5$: $\begin{bmatrix} 1 & 0 & -1 \\ -2 & 2 & -2 \\ -1 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

$\text{rank}(\lambda_1 I - A) = 2 \Rightarrow \text{nullity}(\lambda_1 I - A) = 1$

$(\lambda_1 I - A)\vec{x} = \vec{0}$ ↳ eigenspace dim $= 1$

$\Rightarrow$ 1 eigenvector

$\lambda_2 = 3$ $\begin{bmatrix} -1 & 0 & -1 \\ -2 & 0 & -2 \\ -1 & 0 & -1 \end{bmatrix}$ rank is 1

nullity is 2

$\lambda_2 = 3$ has 2 eigenvectors. geom. mult.

Total eigenvectors: $\lambda_1 = 5$ $\lambda_2 = 3$

$1 + 2 = 3$

We have enough lin. indep. eigenvectors to construct

$P = [\vec{P}_1 \ \vec{P}_2 \ \vec{P}_3]$, invertible.

So A is diagonalizable.

eigenvectors

If λ_0 is an eigenvalue of an $n \times n$ matrix A, then the dimension of the eigenspace corresponding to λ_0 is called the **geometric multiplicity** of λ_0, and the number of times that $\lambda - \lambda_0$ appears as a factor in the characteristic polynomial of A is called the **algebraic multiplicity** of λ_0.

$(\lambda-3)^2(\lambda-5)=0$

Theorem 5.2.4 Geometric and Algebraic Multiplicity
If A is a square matrix, then:
a) For every eigenvalue of A, the geometric multiplicity is less than or equal to the algebraic multiplicity.
b) A is diagonalizable if and only if its characteristic polynomial can be expressed as a product of linear factors, and the geometric multiplicity of every eigenvalue is equal to the algebraic multiplicity.

#14 Find the geometric and algebraic multiplicity of each eigenvalue of the matrix A, and determine whether A is diagonalizable. If A is diagonalizable, then find a matrix P that diagonalizes A, and find $P^{-1}AP$.

$$A = \begin{bmatrix} 5 & 0 & 0 \\ 1 & 5 & 0 \\ 0 & 1 & 5 \end{bmatrix}$$

$(\lambda-5)^3 = 0 \Rightarrow \lambda = 5$ (alg. mult. 3)

$\lambda I - A = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$ has rank 2

and nullity 1. Geom. mult. of $\lambda=5$ is 1.

A is not diagonalizable.

Theorem 5.2.2 $\lambda = 2,3,4$ / $\lambda = 3,5$
a) If $\lambda_1, \lambda_2, \ldots, \lambda_k$ are distinct eigenvalues of a matrix A, and if $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are corresponding eigenvectors, then $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a linearly independent set.
b) An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

3x3: $(\lambda-3)(\lambda-4)(\lambda-5)=0$ guaranteed diagonalizable

$(\lambda-3)^2(\lambda-5)=0$ maybe, maybe not.

$P^{-1}AP = D$ diagonalizes A

Similarity

not necessarily diagonal

Definition: A transformation of the form $A \to P^{-1}AP$ is called a **similarity transformation**.

Definition: If A and B are square matrices, then we say that **B is similar to A** (or that A and B are **similar matrices**) if there is an invertible matrix P such that $B = P^{-1}AP$.

Example: Prove that if A and B are similar matrices, then $\det(A) = \det(B)$.

pf: Suppose A & B are similar. Then $\exists$ $P, P^{-1} \ni B = P^{-1}AP \Rightarrow \det(B) = \det(P^{-1}AP)$

$\det(B) = \det(P^{-1})\det(A)\det(P) = \frac{1}{\det(P)}\det(A)\det(P)$

$\det(B) = \det(A)$ ✓

Any property that is preserved by a similarity transformation is called a **similarity invariant** and is said to be **invariant under similarity**.

Similarity invariants

The following are the same for A and $P^{-1}AP$:

- Determinant
- Invertibility
- Rank
- Nullity
- Trace
- Characteristic polynomial
- Eigenvalues
- Eigenspace dimension

$B = P^{-1}AP$: $A\vec{x} = \lambda\vec{x}$ $B\vec{x} = \lambda\vec{x}$

similar — same — can be different